ISI Type-B Mock Test

Answer as much as you can. Best of Luck!

1. Prove that among 7 real numbers $y_1, y_2, \ldots y_7$ there exists at least two real numbers a and b such that; $\frac{a-b}{1+ab} \leq \frac{1}{\sqrt{3}}$.
   Hint: What does the expression $\frac{a-b}{1+ab}$ remind you of? Try substituting accordingly. **(10 marks)**

2. Find the number of quadratic polynomials $ax^2 + bx + c$, which satisfy the following conditions:

   (i) a, b, c are distinct.

   (ii) $a, b, c \in \{1, 2, 3, \ldots 1999\}$

   (iii) $(x + 1)$ divides $ax^2 + bx + c$

   (12 marks)

3. Find all natural numbers n such that $n^2 - 19n + 99$ is a perfect square. **(8 marks)**

4. Suppose $f : [0, 1] \to \mathbb{R}$ is a continuous and differentiable function such that, $f(0) = 0$ and $0 \leq f'(x) \leq 2f(x)$. Show that, $f(x) = 0$ for all $x \in [0, 1]$. **(10 marks)**

5. A differentiable function f satisfies $f(1) = 2, f(2) = 3$ and $f(3) = 1$. Show that, $f'(x) = 0$ for some $x \in (1, 3)$. **(8 marks)**

6. Find all integers p, q, r, s such that, $p + qrs = q + prs = r + pqs = s + pqr = 2$. **(12 marks)**

7. Prove that, in $\triangle ABC$, if the angles are denoted by A, B and C, then;

$$\sin A + \sin B + \sin C \leq \cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2}$$

   (10 marks)